Change of Variables Multivariable Calculus X. Du

- Useful to change the coordinate system to have simpler bounds when the current bounds are difficult, if impossible to describe and evaluate in the standard Cartesian system.
- Example: transform a region in the xy-plane to the uv-plane (i.e. uv-substitution).
- Generally, transform $(x_1, x_2, ..., x_n)$ to $(u_1, u_2, ..., u_n)$.
- Application of the **Implicit Function Theorem**.
- Come up with a vector-valued function that takes in a vector in one coordinate system and transform it to vector in another coordinate system.
- Don't forget to add the Jacobian! The Jacobian is what allows us to change the coordinate system of the integral - it is what makes change of variables possible!
- Example of change in two variables (can be generalized to *n* variables):

$$\iint_D f(x,y)\,dy\,dx = \iint_S f(u(x,y), v(x,y)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du$$

  - Don't forget to take the absolute value of the Jacobian!
- Common transformations:
  - Cartesian plane to polar plane: $(x,y) \leftrightarrow (r,\theta)$
    - $x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2 \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$
  - Cartesian to cylindrical: $(x,y,z) \leftrightarrow (r,\theta,z)$: same as Cartesian to polar.
  - Cartesian to spherical: $(x,y,z) \leftrightarrow (\rho,\theta,\varphi)$
    - $x = \rho\cos\theta\sin\varphi \qquad y = \rho\sin\theta\sin\varphi \qquad z = \rho\cos\varphi$
    - $\rho^2 = x^2 + y^2 + z^2 \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \varphi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$
  - Jacobians for these common transformations:
    - For $(x,y) \leftrightarrow (r,\theta)$:
      - $\frac{\partial(x,y)}{\partial(r,\theta)} = r \qquad \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2}}$
    - For $(x,y,z) \leftrightarrow (r,\theta,z)$:
      - $\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r \qquad \frac{\partial(r,\theta,z)}{\partial(x,y,z)} = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2}}$
    - For $(x,y,z) \leftrightarrow (\rho,\theta,\varphi)$:
      - $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \rho^2\sin\varphi \qquad \frac{\partial(r,\theta,\varphi)}{\partial(x,y,z)} = \frac{1}{\rho^2\sin\varphi}$